Remarks on Murre's conjecture on Chow groups*

by

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Abstract.

For certain product varieties, Murre's conjecture on Chow groups is investigated. In particular, it is proved that Murre's conjecture (B) is true for two kinds of fourfolds. Precisely, if C is a curve and X is an elliptic modular threefold over k (an algebraically closed field of characteristic 0) or an abelian variety of dimension 3, then Murre's conjecture (B) is true for the fourfold $X \times C$.

Key Words: motivic decomposition, Chow group, curve, abelian variety, elliptic modular threefold

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1. Introduction

We will work with the category $\mathscr{V}_k$ of smooth projective varieties over a field k. Let $X \in \mathscr{V}_k$ be irreducible and of dimension d. Let $H(X) := H^*_{et}(\overline{X}, \mathbb{Q}_l)$ be the l-adic cohomology groups over a (fixed) algebraic closure $\overline{k}$ of k, where $\overline{X} = X \times_k \operatorname{Spec}(\overline{k})$ and $l \neq \operatorname{ch}(k)$ is a prime, and let $\operatorname{cl}_X : Z^i(X) \to H^{2i}(X)$ be the cycle map associated to $H(X)$, where $Z^i(X)$ is the group of algebraic cycles of codimension i of X. We have the well-known Künneth formula:

$$H^{2d}(X \times X) \simeq \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X).$$

Let $\Delta_X \subseteq X \times X$ be the diagonal. Then $\operatorname{cl}_{X\times X}(\Delta_X)$ has the Künneth decomposition:

$$\operatorname{cl}_{X\times X}(\Delta_X) = \pi_0^{\text{hom}} + \pi_1^{\text{hom}} + \ldots + \pi_{2d}^{\text{hom}},$$

where $\pi_i^{\text{hom}} \in H^{2d-i}(X) \otimes H^i(X)$ is the i-th Künneth component.

Let $A^j_{\text{num}}(X)$ (resp. $A^j_{\text{rat}}(X) = \operatorname{CH}^j(X)$) be the groups of algebraic cycles of codimension j modulo the numerical equivalence (resp. rational equivalence). Grothendieck's Lefschetz standard conjecture implies the π_i^{hom} are all algebraic (i.e., they are all in the image of the cycle map). Assuming additionally the conjecture that the homological equivalence coincides with the numerical equivalence ([13]), the diagonal (modulo the numerical equivalence) has a canonical decomposition into a sum of orthogonal idempotents (also called projectors)

$$\Delta_X = \pi_0^{\text{num}} + \pi_1^{\text{num}} + \ldots + \pi_{2d}^{\text{num}}. \tag{1}$$

in the correspondence ring $A^d_{\text{num}}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, in the category of Grothendieck motives $\mathscr{M}_k^{\text{num}}$ ([13]) (w.r.t. the numerical equivalence), the motive $h(X) \in \mathscr{M}_k^{\text{num}}$ has a canonical decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus \ldots \oplus h^{2d}(X), \tag{2}$$

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where $h^i(X) := h(X, \pi_i^{\mathrm{num}}, 0) \in \mathscr{M}_k^{\mathrm{num}}$ (See [13] for details).

Furthermore, Murre ([15]) expected that the conjectural decomposition (1) exists even in $\mathrm{CH}^d(X \times X; \mathbb{Q}) := \mathrm{CH}^d(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q} := A^d_{\mathrm{rat}}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and hence in the category of Chow motives $\mathscr{M}_k^{\mathrm{rat}}$ (w.r.t. the rational equivalence), $h(X) \in \mathscr{M}_k^{\mathrm{rat}}$ has a decomposition as in (2). In this new setting, the decomposition is not canonical any more. However, from this conjectural decomposition, Murre ([15]) conjectured a very interesting filtration on rational Chow groups which relates the rational equivalence to the homological equivalence in finite steps as done by the conjectural Bloch-Beilinson filatration.

More precisely, as in [15], we will say that X has a *Chow-Künneth decomposition* over k if there exist $\pi_i \in \mathrm{CH}^d(X \times X; \mathbb{Q})$, $0 \le i \le 2d$, satisfying

(i) π_i are mutually orthogonal projectors;

(ii) $\sum_i \pi_i = \mathrm{cl}_{X\times X}(\Delta_X)$;

(iii) $\mathrm{cl}_{X\times X}(\pi_i) = \pi_i^{\mathrm{hom}}$ (the i-th Künneth component).

Equivalently, the Chow motive of X has a (Chow-Künneth) decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus \ldots \oplus h^{2d}(X),$$

where $h^i(X) := h(X, \pi_i, 0) \in \mathscr{M}_k^{\mathrm{rat}}$.

Then, Murre proposed in [15] the following famous conjecture.

Murre's Conjecture

(A): There exists a Chow-Künneth decomposition for every irreducible variety $X \in \mathscr{V}_k$ of dimension d.

(B): $\pi_0, \ldots, \pi_{j-1}$ and $\pi_{2j+1}, \ldots, \pi_{2d}$ act as zero on $\mathrm{CH}^j(X; \mathbb{Q})$.

(C): Let $F^v\mathrm{CH}^j(X;\mathbb{Q}) = \mathrm{Ker}\pi_{2j} \bigcap \mathrm{Ker}\pi_{2j-1} \bigcap ... \bigcap \mathrm{Ker}\pi_{2j-v+1}$. Then the filtration $F^\bullet$ is independent of of the ambiguity in the choice of the π_i.

(D): $F^1\mathrm{CH}^j(X;\mathbb{Q}) = \mathrm{CH}^j_{\mathrm{hom}}(X;\mathbb{Q}) := \mathrm{Ker}(\mathrm{cl}_X)$.

It was shown by Jannsen ([9]) that Murre's conjecture is equivalent to the Bloch-Beilinson conjecture on rational Chow groups and the two conjectural filtrations proposed respectively by Murre, and Bloch and Beilinson coincides. The main advantage of Murre's conjecture over Bloch-Beilinson's is that one can check the statements for specific varieties as we will do in this paper.

Until now, Murre's conjecture is verified for only a few special varieties. It is known that (A) is true for curves, surfaces ([14]), Abelian varieties ([17][4]), Brauer-Severi varieties, some threefold ([2][3]), some special fourfold ([11]), certain modular varieties ([6][7]) and varieties whose Chow motives are finite-dimensional ([10]). As for the other parts of Murre's conjecture, it is known that (B) and (D) are true for the product of a curve and a surface ([15]), that (B) is true for the product of two surfaces ([11]) and that some part of (D) is true for the product of two surfaces ([11][12]). Jannsen ([10]) proved that (A), (B), (C) and (D) are true for some very special higher dimensional varieties over some special ground fields, in particular, he proved that if k is a rational or elliptic function field (in one variable) over a finite field $\mathbb{F}$ and X_0 is an arbitrary product of rational and elliptic curves over $\mathbb{F}$, then (A)-(D) hold for $X_0 \times_{\mathbb{F}} k$. Gordon and Murre ([8]) proved that (A)-(D) are true for elliptic modular threefold over a field of characteristic 0.

In this paper, we consider Murre's conjecture for certain product varieties. Concretely, we consider such a problem: if the conjecture is true for X, when is it also true for the product of X with a curve or some other variety ? In section 2, we consider the case of the product of a variety with a projective space. In section 3, we consider the case of the product of a variety with a curve. In particular, we generalize Murre's discussion given in [16], and as consequences, we prove that if C is a (smooth projective connected) curve, then Murre's conjecture (B) is true for $X \times C$, where X is an elliptic modular threefold over k (an algebraically closed field of characteristic 0) or X is an abelian variety of

dimension 3. This implies particularly that (B) is true for two new kinds of fourfolds other than products of two surfaces considered in [11] and [12].

2. Products with projective spaces

Fix a field k. Let X (resp. C) be a smooth projective irreducible variety (resp. curve) over k. Let X be of dimension d. In the following, we will always denote by $Z \in \mathrm{CH}^j(X)$ a cycle class. In addition, we denote by p with some lower indices the projection from a product variety to the corresponding factors.

In the proof of Theorem 2.3, the following lemma is crucial.

Lemma 2.1 ([5]) *Let $\mathcal{E}$ be a vector bundle of rank $r = e+1$ on a scheme X of finite type over* $\mathrm{Spec}(k)$, *with the projection $\pi : \mathcal{E} \to X$. Let $\mathbb{P}(\mathcal{E})$ be the associated projective bundle, p the projection from $\mathbb{P}(\mathcal{E})$ to X, and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ the tautological line bundle on $\mathbb{P}(\mathcal{E})$. Then there are canonical isomorphisms*

$$\bigoplus_{i=0}^{e} \mathrm{CH}^{j-i}(X) \longrightarrow \mathrm{CH}^j(\mathbb{P}(\mathcal{E}))$$

$$(\alpha_i) \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^i \cap p^*\alpha_i,$$

where $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ is the first Chern class.

□

Applying Proposition 3.1 in [5], it is easy to show that the inverse of the map in Lemma 2.1 is the map

$$\mathrm{CH}^j(\mathbb{P}(\mathcal{E})) \longrightarrow \bigoplus_{i=0}^{e} \mathrm{CH}^{j-i}(X),\ \beta \mapsto (\beta_i),$$

where $\beta_e = p_*\beta$ and for $0 \le i \le e-1$,

$$\beta_i = p_*(c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^{e-i} \cap \beta - \sum_{t=1}^{e-i} c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^{e+t} \cap p^*\beta_{i+t}).$$

Lemma 2.2 ([15]) *Assume that Y_i $(i = 1, 2)$ are smooth projective irreducible varieties over k. Let $Y = Y_1 \times Y_2$. If Y_i $(i = 1, 2)$ has a Chow-Künneth decomposition, then Y has also a Chow-Künneth decomposition.*

□

Theorem 2.3 *Let X be a smooth projective irreducible variety of dimension d over k. If (A),(B) and (D) are true for X, then they are also true for $X \times \mathbb{P}^r$.*

Proof. Let $c = c_1(\mathcal{O}_{\mathbb{P}^r}(1))$, that is, the class of any hyperplane in $\mathbb{P}^r$. For each $0 \le i \le r$, set

$$\pi_{2i} = c^{r-i} \times c^i, \quad \pi_{2i+1} = 0, \quad h^i(\mathbb{P}^r) = (\mathbb{P}^r, \pi_i).$$

Then it is easy to see that there is the Chow-Künneth decomposition

$$h(\mathbb{P}^r) = \bigoplus_{i=0}^{2r} h^i(\mathbb{P}^r) = \bigoplus_{t=0}^{r} h^{2t}(\mathbb{P}^r).$$

Assume that X has the Chow-Künneth decomposition

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X), \quad h^i(X) = (X, \pi'_i).$$

Then, $X \times \mathbb{P}^r$ has a Chow-Künneth decomposition:

$$h(X \times \mathbb{P}^r) = \bigoplus_{m=0}^{2(d+r)} h^m(X \times \mathbb{P}^r, \pi_m),$$

where $\pi_m = \sum_{p+2q=m} \tau_*(\pi'_p \times c^{r-q} \times c^q)$.

On the other hand, from Lemma 2.1, we have the isomorphisms:

$$\phi : \mathrm{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}) \to \bigoplus_{i=0}^{r} \mathrm{CH}^{j-i}(X; \mathbb{Q}),\ Z \longmapsto (Z_i),$$

where $Z_i = p_{1*}(([X] \times c^{r-i}) \cdot Z)$ with

$$\phi^{-1}((Z_i)) = \sum_{i=0}^{r} ([X] \times c^i) \cdot p_1^* Z_i = \sum_{i=0}^{r} Z_i \times c^i,$$

and

$$\varphi : \mathrm{CH}^{d+r}(X \times \mathbb{P}^r \times X \times \mathbb{P}^r; \mathbb{Q}) \longrightarrow \bigoplus_{i=0}^{r} \bigoplus_{t=0}^{r} \mathrm{CH}^{d+r-i-t}(X \times X; \mathbb{Q}),$$

$$\alpha \longmapsto (\alpha_{it})$$

where $\alpha_{it} = p_{12*}(([X \times X] \times c^{r-t} \times c^{r-i}) \cdot \tau^* \alpha)$. In fact, we have

$$\begin{aligned} \alpha_{it} &= p_{13*}(([X] \times c^{r-t} \times [X]) \cdot p_{123*}(([X \times \mathbb{P}^r \times X] \times c^{r-i}) \cdot \alpha)) \\ &= p_{13*}(([X] \times c^{r-t} \times [X] \times c^{r-i}) \cdot \alpha) \\ &= p_{12*}(([X \times X] \times c^{r-t} \times c^{r-i}) \cdot \tau^* \alpha), \end{aligned}$$

where τ is the isomorphism exchanging the second and the third factor of the product variety $X \times X \times \mathbb{P}^r \times \mathbb{P}^r$. Clearly, we also have $\varphi^{-1}((\alpha_{it})) = \sum_{i,t} \tau_*(\alpha_{it} \times c^t \times c^i)$.

Now, define the map

$$\Phi : \mathrm{CH}^{d+r}(X \times \mathbb{P}^r \times X \times \mathbb{P}^r; \mathbb{Q}) \times \mathrm{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}) \longrightarrow \mathrm{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}),$$

$$\Phi(\alpha, Z) := \alpha(Z) := p_{34*}(\alpha \cdot (Z \times [X \times \mathbb{P}^r])).$$

So we have the following diagram

$$\begin{tikzcd} \mathrm{CH}^{d+r}(X \times \mathbb{P}^r \times X \times \mathbb{P}^r; \mathbb{Q}) \times \mathrm{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}) \arrow[r, "\Phi"] \arrow[d, "\varphi \times \phi"'] & \mathrm{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}) \arrow[d, "\phi"] \\ \bigoplus_{i=0}^r \bigoplus_{t=0}^r \mathrm{CH}^{d+r-i-t}(X \times X; \mathbb{Q}) \times \bigoplus_{i=0}^r \mathrm{CH}^{j-i}(X; \mathbb{Q}) \arrow[r] & \bigoplus_{i=0}^r \mathrm{CH}^{j-i}(X; \mathbb{Q}) \end{tikzcd}$$

Here the lower arrow is defined by the other three.

Note that if $t + i = r$, then we have

$$\begin{aligned} \tau_*(\pi'_p \times c^t \times c^q) \cdot (Z_i \times c^i) &= p_{34*}(\tau_*(\pi'_p \times c^t \times c^q) \cdot (Z_i \times c^i \times [X \times \mathbb{P}^r])) \\ &= p_{34*}\tau_*((\pi'_p \times c^t \times c^q) \cdot (Z_i \times [X] \times c^i \times [\mathbb{P}^r])) \\ &= p_{24*}((\pi'_p \cdot (Z_i \times [X])) \times c^{t+i} \times c^q)) \\ &= p_{2*}((\pi'_p \cdot (Z_i \times [X])) \times c^{t+i}) \times c^q \\ &= \pi'_p(Z_i) \times c^q. \end{aligned}$$

So we conclude that

$$\tau_*(\pi'_p \times c^t \times c^q) \cdot (Z_i \times c^i) = \begin{cases} \pi'_p(Z_i) \times c^q, & \text{if } t + i = r; \\ 0\,, & \text{otherwise.} \end{cases}$$

Hence, we can translate the projectors on $X \times \mathbb{P}^r$ to those on X as follows.

$$\begin{aligned}\phi(\pi_m(Z)) &= \phi \cdot \Phi \cdot (\varphi \times \phi)^{-1}(\varphi(\pi_m), (Z_i)) = \phi \cdot \Phi(\pi_m, \sum_{l=0}^{r} Z_i \times c^i) \\ &= \sum_{i=0}^{r} \sum_{p+2q=m} \phi(\tau_*(\pi'_p \times c^{r-q} \times c^q) \cdot (Z_i \times c^i)) \\ &= \sum_{i=0}^{r} \sum_{p+2q=m, i=q} \phi(\pi'_p(Z_i) \times c^q) \\ &= (\pi'_m(Z_0), \pi'_{m-2}(Z_i), \ldots, \pi'_{m-2r}(Z_r))\end{aligned}$$

Now, we can prove that the conjectures are true for $X \times \mathbb{P}^r$.

For (B), from $0 \leq m \leq j-1$ we have $m - 2i \leq (j-i) - 1$. If $2j+1 \leq m \leq 2(d+r)$, then

$$2(j-i) + 1 \leq m - 2i \Longleftrightarrow 2j + 1 \leq m.$$

So, by the assumptions on X, we see that

$$\pi_m(Z) = 0, \quad \text{for} \ \ 0 \leq m \leq j-1 \ \ \text{or} \ \ 2j+1 \leq m.$$

For (D), suppose that $Z \in \mathrm{CH}^j_{\mathrm{hom}}(X \times \mathbb{P}^r; \mathbb{Q}) = \mathrm{Ker}(\mathrm{cl}_{X \times \mathbb{P}^r})$. Then since $Z_i \in \mathrm{Ker}(\mathrm{cl}_X) = \mathrm{CH}^{j-i}_{\mathrm{hom}}(X; \mathbb{Q}) = \mathrm{Ker}(\pi'_{2(j-i)})$ by assumption, we have $Z \in \mathrm{Ker}(\pi_{2j})$. This completes the proof of Theorem 2.3.

□

Remark 2.4 (i) We expect that Theorem 2.3 is also true for non-trivial projective bundles.

(ii) For (C), we can say nothing yet since from the projectors on X we can get only one but not all set of projectors on $X \times \mathbb{P}^r$.

Corollary 2.5 *Let S_1, S_2 be smooth projective surfaces over k. Then conjectures (A) and (B) are true for $S_1 \times S_2 \times \mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_n}$.*

Proof: From Lemma 2.2 and the main theorem of [14], we know that conjectures (A) and (B) are true for $S_1 \times S_2$, so the result follows from Theorem 2.3. □

3. Products with curves

Let C be a smooth projective curve over a field k and $e \in C(k)$. It is well-known (see [18] for details) that C has the Chow-Künneth decomposition

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C),$$

where $h^i(C) = (C, \pi''_i)$ with

$$\pi''_0 = [e \times C], \quad \pi''_2 = [C \times e], \quad \pi''_1 = \Delta_C - \pi''_0 - \pi''_2.$$

Assume that the irreducible variety $X \in \mathscr{V}_k$ has the Chow-Künneth decomposition

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X), \quad h^i(X) = (X, \pi'_i).$$

Then, the product variety $X \times C$ has the Chow-Künneth decomposition

$$h(X \times C) = \bigoplus_{m=0}^{2(d+1)} h^m(X \times C, \pi_m),$$

where, explicitely,

$\pi_0 = \pi'_0 \times [e \times C]$,

$$\pi_1 = \pi_1' \times [e \times C] + \pi_0' \times (\Delta_C - [e \times C] - [C \times e]),$$
$$\pi_m = \pi_m' \times [e \times C] + \pi_{m-1}' \times (\Delta_C - [e \times C] - [C \times e]) + \pi_{m-2}' \times [C \times e],\ m \geq 2.$$

Let
$$\mathrm{CH}^j_{\mathrm{alg}}(X;\mathbb{Q}) := \{Z \in \mathrm{CH}^j(X;\mathbb{Q}) : Z \sim_{\mathrm{alg}} 0\},$$
where $Z \sim_{\mathrm{alg}} 0$ means that Z is algebraically equivalent to 0.

In the proof of Theorem 3.3, we need the following computations.

Lemma 3.1 *For any $Z \in CH^j(X \times C;\mathbb{Q})$, we have*
(i) $(\pi_m' \times [e \times C])(Z) = \pi_m'(Z(e)) \times [C]$;
(ii) $(\pi_m' \times [C \times e])(Z) = \pi_m'(p_{1*}Z) \times [e]$.

Proof: (i) We have
$$\begin{aligned}(\pi_m' \times [e \times C])(Z) &= p_{34*}(\tau_*(\pi_m' \times [e \times C]) \cdot (Z \times [X \times C]))\\ &= p_{34*}(\tau_*(\pi_m' \times [e]) \cdot (Z \times [X]) \times [C]\\ &= p_{3*}(\tau_*(\pi_m' \times [e]) \cdot (Z \times [X]) \times [C]\\ &= \pi_m'(Z(e)) \times [C],\end{aligned}$$
where $Z(e) = p_{1*}(Z \cdot (X \times [e]))$. Note that in the last equality, we have used the following computation.
$$\begin{aligned}p_{3*}(\tau_*(\pi_m' \times [e]) \cdot (Z \times [X]) &= p_{3*}(\tau_*((\pi_m' \times [C]) \cdot [X \times \times [e] \times X) \cdot (Z \times [X])\\ &= p_{3*}((p_{13}^*\pi_m' \cdot [X \times e \times X]) \cdot (Z \times [X]))\\ &= p_{3*}((p_{13}^*\pi_m' \cdot ((Z \cdot [X \times e]) \times [X]))\\ &= p_{2*}p_{13*}((p_{13}^*\pi_m' \cdot ((Z \cdot [X \times e]) \times [X]))\\ &= p_{2*}(\pi_m' \cdot p_{13*}((Z \cdot [X \times e]) \times [X]))\\ &= p_{2*}(\pi_m' \cdot (p_{13*}(Z \cdot [X \times e]) \times [X]))\\ &= \pi_m'(Z(e)).\end{aligned}$$

(ii) Similar to (i), we have
$$\begin{aligned}(\pi_m' \times [C \times e])(Z) &= p_{34*}(\tau_*(\pi_m' \times [C \times e]) \cdot (Z \times [X \times C]))\\ &= p_{34*}(\tau_*(\pi_m' \times [C]) \cdot (Z \times [X])) \times [e])\\ &= p_{3*}((\tau_*(\pi_m' \times [C]) \cdot (Z \times [X])) \times [e]\\ &= \pi_m'(p_{1*}Z) \times [e].\end{aligned}$$

□

Lemma 3.2 *Let $Z \in CH^j(X \times C;\mathbb{Q})$. Then*
(i) $(\pi_m' \times \Delta_C)(Z) = p_{23*}(p_{13}^*Z \cdot (\pi_m' \times [C]))$;
(ii) $(\mathrm{id}_X \times f)^*((\pi_m' \times \Delta_C)(Z)) = (\pi_m')_K((\mathrm{id}_X \times f)^*Z)$, *where $K = k(C)$ is the function field of C, $f : Spec(K) \longrightarrow C$ is the natural morphism and $(\pi_m')_K = \pi_m' \times \Delta_{Spec(K)}$.*

Proof: (i) Let $\delta_C : C \longrightarrow C \times C$ be diagonal morphism. Then, we have
$$\begin{aligned}(\pi_m' \times \Delta_C)(Z) &= p_{34*}(\tau_*(\pi_m' \times \Delta_C) \cdot (Z \times [X \times C]))\\ &= p_{34*}(\tau_*(\mathrm{id}_{X\times X} \times \delta_C)_*(\pi_m' \times \Delta_C) \cdot (Z \times [X \times C]))\\ &= p_{23*}((\pi_m' \times \Delta_C) \cdot (\mathrm{id}_{X\times X} \times \delta_C)^*\tau^*Z))\\ &= p_{23*}(p_{13}^*Z \cdot (\pi_m' \times [C])).\end{aligned}$$

(ii) From (i) and the following diagram
$$\begin{tikzcd} X_K \times_K X_K \arrow[r, "p_{X_K}"] \arrow[d, "\mathrm{id}_X \times \mathrm{id}_X \times f"'] & X_K \arrow[d, "\mathrm{id}_X \times f"] \\ X \times X \times C \arrow[r, "p_{23}"] & X \times C \end{tikzcd}$$

we have

$$\begin{aligned}(\mathrm{id}_X \times f)^*((\pi'_m \times \Delta_C)(Z)) &= (\mathrm{id}_X \times f)^* p_{23*}(p_{13}^* Z \cdot \pi'_m \times [C]) \\ &= p_{X_K*}(\mathrm{id}_X \times \mathrm{id}_X \times f)^*(p_{13}^* Z \cdot \pi'_m \times [C]) \\ &= p_{X_K*}((\mathrm{id}_X \times \mathrm{id}_X \times f)^* p_{13}^* Z \cdot (\mathrm{id}_X \times \mathrm{id}_X \times f)^*(\pi'_m \times [C])) \\ &= p_{X_K*}(((\mathrm{id}_X \times f)^* Z \times_K X_K) \cdot \pi'_m \times \Delta_{\mathrm{spec}K}) \\ &= (\pi'_m \times \Delta_{\mathrm{spec}K})((\mathrm{id}_X \times f)^* Z) \\ &= (\pi'_m)_K((\mathrm{id}_X \times f)^* Z).\end{aligned}$$

□

Our main theorem is the following

Theorem 3.3 *Let k be an algebraically closed field, $X \in \mathscr{V}(k)$ and $C \in \mathscr{V}(k)$ an irreducible curve with the function field $K = k(C)$. Assume that (A) and (B) are true for X and X_K, and that for any j, $CH^j_{alg}(X_K;\mathbb{Q}) \subseteq Ker((\pi'_{2j})_K)$. Then (A) and (B) are also true for $X \times C$.*

Proof: The statement about (A) is obvious. We will consider (B) in the following. Let $Z \in \mathrm{CH}^j(X \times C;\mathbb{Q})$. Easy computations shows that (B) is true if Z is of the form $Z' \times [C]$ with $Z' \in \mathrm{CH}^j(X;\mathbb{Q})$. So, we can assume $Z(e) = 0$, since we have

$$Z = (Z - Z(e) \times [C]) + Z(e) \times [C],$$

$$(Z - Z(e) \times [C])(e) = Z(e) - Z(e) = 0.$$

Assume that

$$0 \le m \le j-1 \quad \text{or} \quad 2j+1 \le m \le 2(d+1).$$

Then, for $m \ge 1$, we have

$$0 \le m-1 \le (j-1)-1 \;\; \text{or} \;\; 2(j-1)+1 \le m-1,$$

and for $m \ge 2$, we have

$$0 \le m-2 \le (j-1)-1 \;\; \text{or} \;\; 2(j-1)+1 \le m-1.$$

From Lemma 3.1 and the assumptions on X, we have (note that $p_{1*}Z \in \mathrm{CH}^{j-1}(X;\mathbb{Q})$)

$$\begin{aligned}(\pi'_m \times [e \times C])(Z) &= \pi'_m(Z(e)) \times [C] = 0, \\ (\pi'_{m-1} \times [e \times C])(Z) &= \pi'_{m-1}(Z(e)) \times [C] = 0, \\ (\pi'_{m-1} \times [C \times e])(Z) &= \pi'_{m-1}(p_{1*}Z) \times [e] = 0, \\ (\pi'_{m-2} \times [C \times e])(Z) &= \pi'_{m-2}(p_{1*}Z) \times [e] = 0.\end{aligned}$$

So, the problem is reduced to prove

$$(\pi'_{m-1} \times \Delta_C)(Z) = 0, \text{ if } \; 1 \le m \le j-1 \text{ or } \; 2j+1 \le m \le 2(d+1).$$

At first, we show that $(\mathrm{id} \times f)^* Z$ is algebraically equivalent to 0 on X_K. In fact, let η be the generic point of C, that is, $K = k(\eta)$, and let $f_\eta : \mathrm{Spec}(K) \longrightarrow C_K$ be the K-point defined by η. Denote $\eta_K = f_\eta(\mathrm{Spec}K)$. Then we have

$$\begin{aligned}Z_K(\eta_K) &= p_{X_K*}(Z_K \cdot X \times \eta_K) = p_{X_K*}(Z_K \cdot (\mathrm{id}_X \times f_\eta)_*(X \times \mathrm{Spec}K)) \\ &= p_{X_K*}(\mathrm{id}_X \times f_\eta)_*((\mathrm{id}_X \times f_\eta)^* Z_K) = (\mathrm{id}_X \times f_\eta)^* Z_K \\ &= (\mathrm{id}_X \times f_\eta)^* p_{XC}^{XCK*}(Z) = (\mathrm{id}_X \times f)^*(Z).\end{aligned}$$

Similarly, let $g_e : \mathrm{Spec}(K) \longrightarrow C_K$ and $g : \mathrm{Spec}k \longrightarrow C$ be the morphisms both defined by e. Denote $e_K = g_e(\mathrm{Spec}(K))$. Then we have $Z_K(e_K) = ((\mathrm{id}_X \times g)^*(Z))_K = Z(e)_K = 0$.

We claim that

$$(\pi'_{m-1})_K((\mathrm{id} \times f)^* Z) = 0, \quad \text{if} \;\; 1 \le m \le j-1 \;\; \text{or} \;\; 2j+1 \le m \le 2(d+1).$$

In fact, if $m=2j+1$, since $(\mathrm{id}\times f)^*Z\in \mathrm{CH}^j(X_K;\mathbb{Q})$ is algebraically equivalent to 0, from the assumption $\mathrm{CH}^j_{\mathrm{alg}}(X_K;\mathbb{Q})\subseteq \mathrm{Ker}((\pi'_{2j})_K)$ we have

$$(\pi'_{m-1})_K((\mathrm{id}\times f)^*Z)=(\pi'_{2j})_K((\mathrm{id}\times f)^*Z)=0;$$

if $1\le m\le j-1$ or $2j+2\le m\le 2(d+1)$, we have $1\le m-1\le j-2$ or $2j+1\le m-1\le 2d+1$, so from the assumptions on X we get $(\pi'_{m-1})_K((\mathrm{id}\times f)^*Z)=0$ since $(\mathrm{id}\times f)^*Z\in \mathrm{CH}^j(X_K;\mathbb{Q})$.

On the other hand, we have the following well known diagram

$$\begin{tikzcd}
\mathrm{CH}^{j-1}(X\times(C-U);\mathbb{Q}) \arrow[r] & \mathrm{CH}^j(X\times C;\mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{CH}^j(X\times U;\mathbb{Q}) \arrow[r] \arrow[d] & 0 \\
& \mathrm{CH}^j(X_K;\mathbb{Q}) \arrow[r, equal] & \varinjlim_{U\subseteq C}\mathrm{CH}^j(X\times U;\mathbb{Q}) &
\end{tikzcd}$$

where the left vertical map is $z\mapsto(\mathrm{id}\times f)^*z$. So from Lemma 3.2 (ii), we have

$$(\mathrm{id}_X\times f)^*((\pi'_{m-1}\times\Delta_C)(Z))=(\pi'_{m-1})_K((\mathrm{id}\times f)^*Z)=0,$$

hence

$$(\pi'_{m-1}\times\Delta_C)(Z)=\sum_i Z'_i\times a_i,\ \text{ with }\ Z'_i\in \mathrm{CH}^{j-1}(X;\mathbb{Q})\ \text{and}\ \ a_i\in \mathrm{CH}^1(C;\mathbb{Q}).$$

In view of $(\pi'_{m-1}\times\Delta_C)^2=\pi'_{m-1}\times\Delta_C$, we conclude that

$$\begin{aligned}(\pi'_{m-1}\times\Delta_C)(Z)&=\sum_i(\pi'_{m-1}\times\Delta_C)(Z'_i\times a_i)\\ &=\sum_i p_{23*}[p_{13}^*(Z'_i\times a_i)\cdot(\pi'_{m-1}\times[C])]\\ &=\sum_i \pi'_{m-1}(Z'_i)\times a_i=0.\end{aligned}$$

This completes the proof of the theorem.

□

Although the theorem above is restricted, we can deduce several interesting consequences.

Corollary 3.4 *If k is an algebraically closed field of characteristic 0 and X is an elliptic modular threefold over k, then (A) and (B) are true for $X\times C$.*

Proof: It was shown in [8] that Murre's conjecture holds for an elliptic modular threefold over a field of characteristic 0. Obviously, conjecture (D) for X_K implies the assumption of Theorem 3.3. So, the result is an immediate consequence of Theorem 3.3.

□

Corollary 3.5 *Assume that algebraic equivalence and rational equivalence coincide on X. If (A) and (B) are true for X and X_K, then (A) and (B) are also true for $X\times C$.*

□

Remark 3.6 Cellular varieties satisfy the first hypothesis of the corollary.

By [1] (see also [4] and [15]), for an abelian variety X of dimension g over any field k, we have the following decomposition

$$\mathrm{CH}^j(X;\mathbb{Q})=\bigoplus_{s=j-g}^{j}\mathrm{CH}^j_s(X),$$

where

$$\mathrm{CH}^j_s(X):=\{\alpha\in \mathrm{CH}^j(X;\mathbb{Q})|n^*\alpha=n^{2j-s}\alpha,\forall\ n\in\mathbb{Z}\}.$$

Corollary 3.7 *Let X be an abelian variety of dimension at most 4 over an algebraically closed field k. Assume that for any j, $CH_0^j(X_K)\bigcap CH_{alg}^j(X_K;\mathbb{Q})=0$. Then (A) and (B) are true for $X\times C$.*
Proof: It follows from [1] that conjecture (B) is true for an abelian variety of dimension at most 4, equivalently, Beauville's vanishing conjecture holds: $\mathrm{CH}_s^j(X)=0$ if $s<0$. By assumption and the fact that the algebraic equivalence is adequate, we have

$$\mathrm{CH}\,^j_{\mathrm{alg}}(X_K;\mathbb{Q})=\bigoplus_{s=1}^{j}(\mathrm{CH}_s^j(X_K)\cap\mathrm{CH}_{\mathrm{alg}}^j(X_K;\mathbb{Q})).$$

On the other hand, by Lemma 2.5.1 of [15], we see that for any j,

$$\mathrm{Ker}((\pi'_{2j})_K)=\bigoplus_{s=1}^{j}\mathrm{CH}_s^j(X_K).$$

Then we can apply Theorem 3.3 to end the proof.

□

Remark 3.8 The assumption of Corollary 3.7 is a consequence of a conjecture of Beauville: the restricted cycle map $c_0:\mathrm{CH}_0^j(X)\longrightarrow H^{2j}(X)$ is injective for any j.
Corollary 3.9 *Let X be an abelian variety of dimension 3 over an algebraically closed field k. Then* (A) *and* (B)*are true for $X\times C$.*
Proof: By [1], we know that the restricted cycle map

$$c_0:\mathrm{CH}_0^j(X_K)\longrightarrow H^{2j}(X_K)$$

is injective for $j=0,1,g-1,g$ if X is an abelian variety of dimension g. So for any j,

$$\mathrm{CH}_0^j(X_K)\bigcap\mathrm{CH}_{\mathrm{alg}}^j(X_K;\mathbb{Q})\subseteq\mathrm{CH}_0^j(X_K)\bigcap\mathrm{CH}_{\mathrm{hom}}^j(X_K;\mathbb{Q})=0.$$

Hence the result follows from Corollary 3.7.

□

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